QUANTUM FEEDBACK NETWORKS

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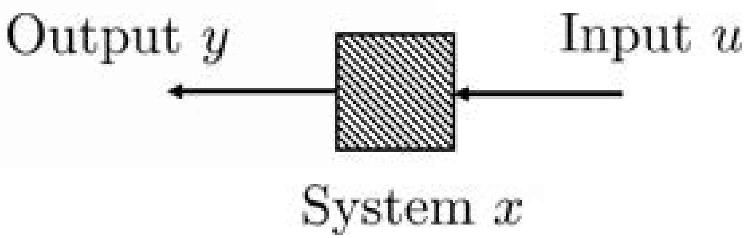
Classical Linear Systems

A model is said to be **linear** if its state dynamics and input-output equations are then assumed to take the following form:

$$\boxed{\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned}}$$



The linear system is by the **model matrix**:

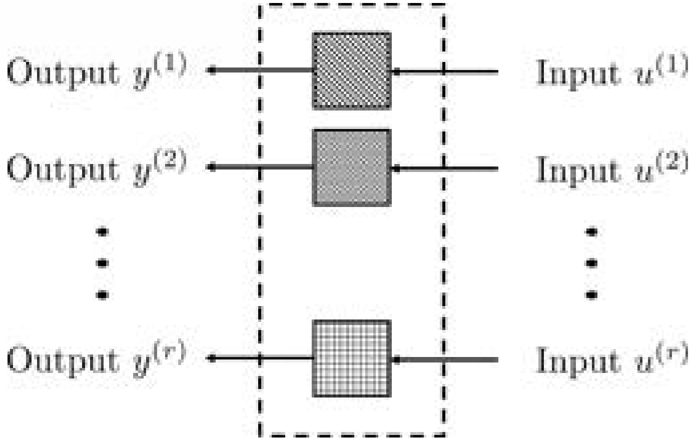
$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{X} \oplus \mathcal{U} \mapsto \mathcal{X} \oplus \mathcal{Y},$$

written in block partition form.

Classical Linear Systems

Given a pair of models $M_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}$ on $(\mathcal{Y}_i, \mathcal{X}, \mathcal{U}_i)$, for $i = 1, 2$, their **superimposition** is the model on $(\mathcal{Y}_1 \oplus \mathcal{Y}_2, \mathcal{X}, \mathcal{U}_1 \oplus \mathcal{U}_2)$ given by

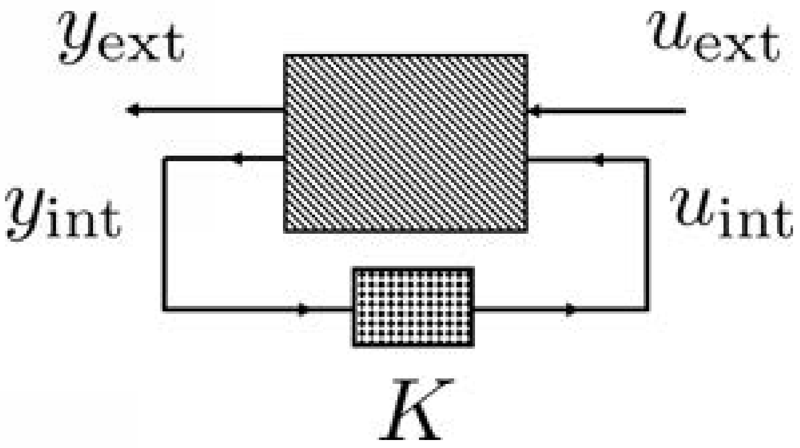
$$M_1 \boxplus M_2 = \begin{bmatrix} A_1 + A_2 & B_1 & B_2 \\ C_1 & D_1 & 0 \\ C_2 & 0 & D_2 \end{bmatrix}.$$



Classical Linear Systems

We begin with an open-loop model on $(\mathcal{Y}_{\mathrm{ext}} \oplus \mathcal{Y}_{\mathrm{int}}, \mathcal{X}, \mathcal{U}_{\mathrm{ext}} \oplus \mathcal{U}_{\mathrm{int}})$ with

$$M = \begin{bmatrix} A & B_{\mathrm{e}} & B_{\mathrm{i}} \\ C_{\mathrm{e}} & D_{\mathrm{ee}} & D_{\mathrm{ei}} \\ C_{\mathrm{i}} & D_{\mathrm{ie}} & D_{\mathrm{ii}} \end{bmatrix}$$



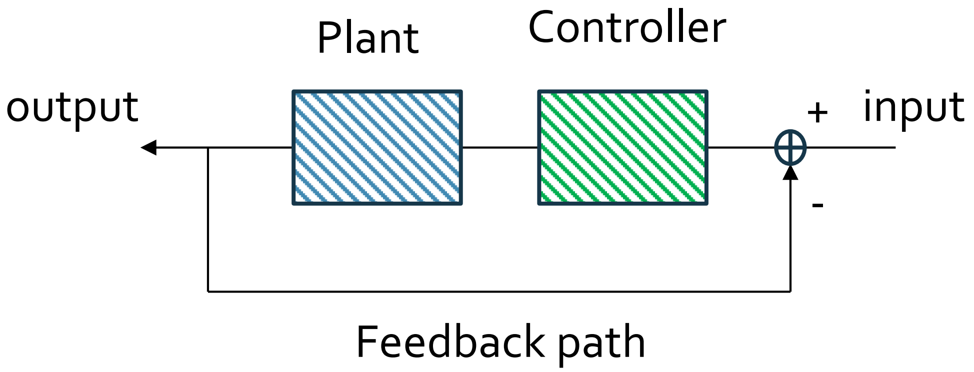
To close the internal loop, we set $y_{\mathrm{i}} = K\, u_{\mathrm{i}}$,

This leads to the **feedback reduction**

$$M_{\mathrm{fb}}(K) = \begin{bmatrix} A & B_{\mathrm{e}} \\ C_{\mathrm{e}} & D_{\mathrm{ee}} \end{bmatrix} + \begin{bmatrix} B_{\mathrm{i}} \\ D_{\mathrm{ei}} \end{bmatrix} (I - K D_{\mathrm{ii}})^{-1} K \begin{bmatrix} C_{\mathrm{i}} & D_{\mathrm{ie}} \end{bmatrix}$$

Well-posedness requires that $(I - K D_{\mathrm{ii}})$ is invertible.

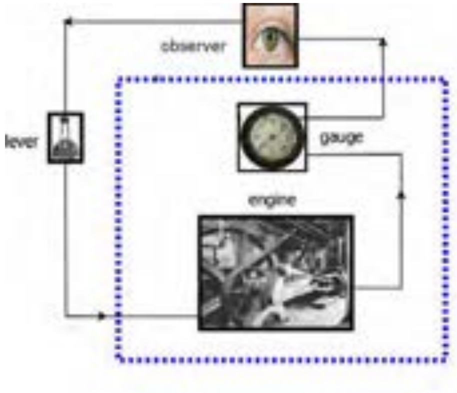
Networks and Feedback Control



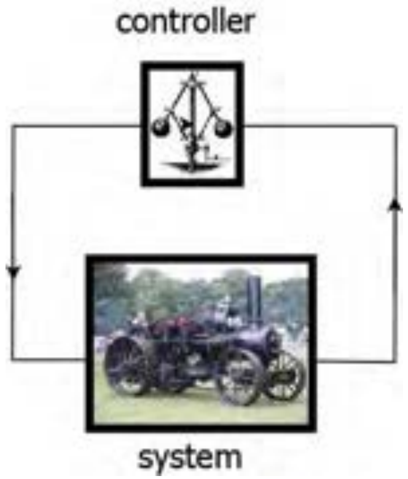
$$Y = GH(U - Y)$$

$$Y = \frac{GH}{1+GH}U$$

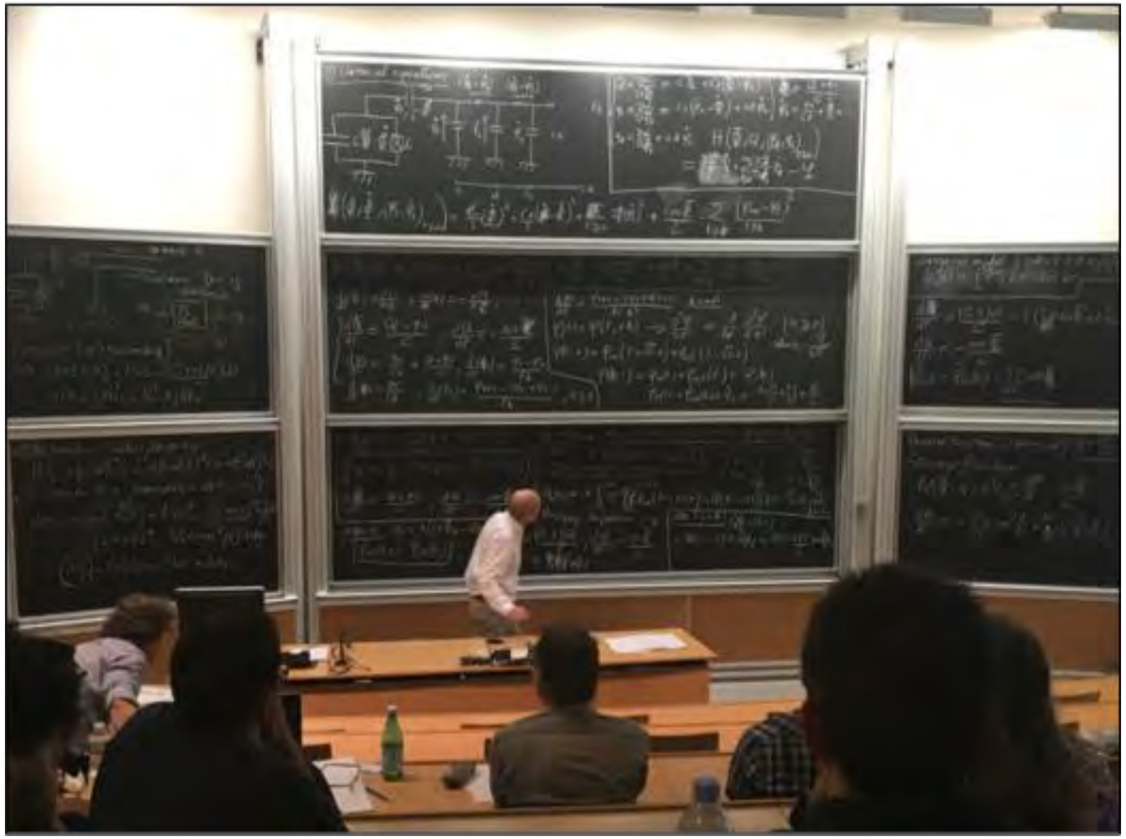
- Measurement Based Feedback Control



- Coherent Feedback Control

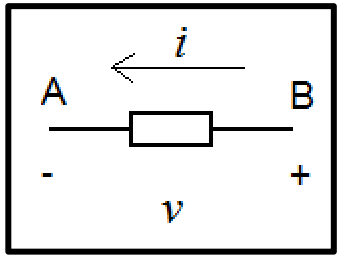


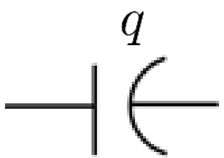
What about circuit QED?



CIRM, 2018

Electric Networks



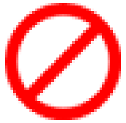
- Capacitors 

$i = \dot{q}, \quad v = \hat{v}(q)$ $\qquad E_C(q) = \int^q \hat{v}(q')dq'$

- Inductors 

$v = \dot{\varphi}, \quad i = \hat{\imath}(\varphi)$ $\qquad E_L(\varphi) = \int^\varphi \hat{\imath}(\varphi')d\varphi'$

- Networks $E = \sum_j E_{C_j}(q_j) + \sum_k E_{L_k}(\varphi_k)$



Capacitor/voltage source loops!
Inductor/current source cutsets!

Chua + Green/MacPherson (1976)

$$\begin{bmatrix} \dot{q}_C \\ \dot{\varphi}_L \end{bmatrix} = \begin{bmatrix} 0 & F^\top \\ -F & 0 \end{bmatrix} \begin{bmatrix} \nabla_C E \\ \nabla_L E \end{bmatrix}$$

F is the fundamental loop matrix

Bernstein and Lieberman (1989)

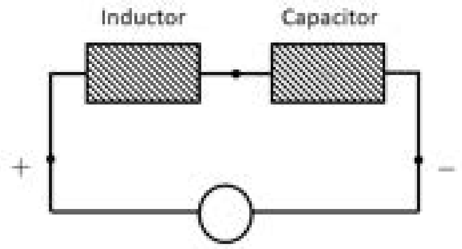
invertible matrices M and N
such that $N^{-1}F\left(M^\top\right)^{-1} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

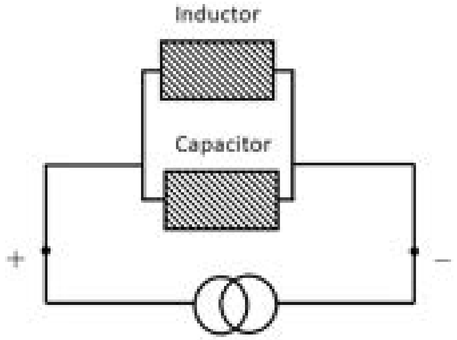
$x = M^{-1}q_C$ and $y = N^{-1}\varphi_C$,
$\tilde{E}(x,y) = E(Mx, Ny)$

Maschke, van der Schaft, Breedveld (1992/5)

Intrinsic Poisson Bracket formulation.
r = active variable, others are casimirs

Series/Parallel





Series

$$L_{\text{ser.}}(q,\dot{q},t) = E_L^*(\dot{q}) - E_C(q) + q v_{\text{in}}(t)\,.$$

$$\frac{\partial L_{\text{ser.}}}{\partial \dot{q}} = \frac{\partial E_L^*(\dot{q})}{\partial \dot{q}} \equiv \tilde{\varphi}_L(\dot{q})\,.$$

$$H_{\text{ser.}}(q_C,\varphi_L,t) = \sup_{\dot{q}}\left\{\dot{q}\varphi_L - L_{\text{ser.}}(q_C,\dot{q},t)\right\}.$$

Parallel

$$L_{\text{par.}}(\varphi,\dot{\varphi},t) = E_C^*(\dot{\varphi}) - E_L(\varphi) + \varphi i_{\text{in}}(t)$$

$$\frac{\partial L_{\text{par.}}}{\partial \dot{\varphi}} = \frac{\partial E_C^*(\dot{\varphi})}{\partial \dot{\varphi}} \equiv \tilde{q}_L(\dot{\varphi})\,.$$

$$H_{\text{par.}}(\varphi_L,q_C,t) = \sup_{\dot{\varphi}}\left\{\dot{\varphi} q_C - L_{\text{par.}}(\varphi_L,\dot{\varphi},t)\right\}.$$

Duality

$$H_{\text{ser.}}(q,\varphi,t) = h(q,\varphi) + q\,v_{\text{in}}(t)$$

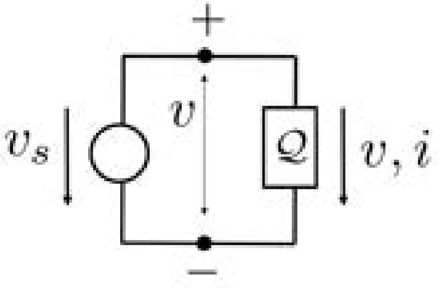
$$H_{\text{par.}}(q,\varphi,t) = h(q,\varphi) + \varphi\,i_{\text{in}}(t)$$

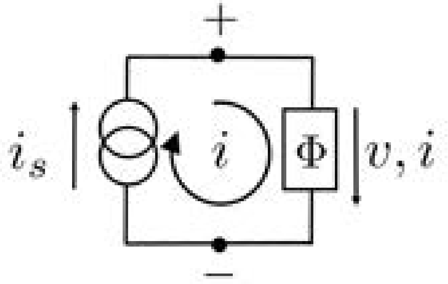
$$h(q,\varphi) = E_C(q) + E_L(\varphi)\,.$$

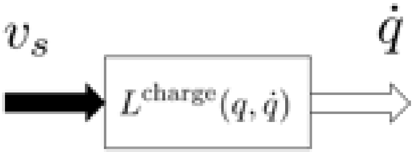
$$\frac{d}{dt}A = \{A, H_{\text{ser.}}\}_{\text{ser.}} \quad \text{and} \quad \frac{d}{dt}A = \{A, H_{\text{par.}}\}_{\text{par.}}$$

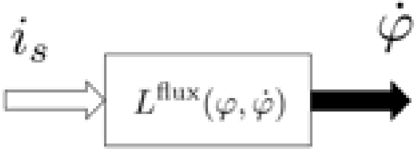
$$\{A,B\}_{\text{ser.}} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial \varphi} - \frac{\partial B}{\partial q}\frac{\partial A}{\partial \varphi} = -\{A,B\}_{\text{par.}}$$

Driven Components









Charge Hamiltonian

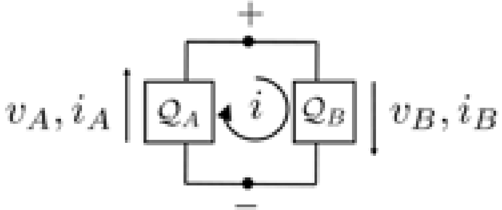
$$H^{\mathrm{c}}\left(q, \hat{\varphi}; v_s\left(t\right)\right) = \sup_{\dot{q}} \left\{\dot{q}\hat{\varphi} - L^{\mathrm{c}}\left(q, \dot{q}; v_s\left(t\right)\right)\right\} \equiv H^{\mathrm{c}}\left(q, \hat{\varphi} + \varphi_s\left(t\right)\right)$$

Flux Hamiltonian

$$H^{\mathrm{f}}\left(\varphi, \hat{q}; i_s\left(t\right)\right) = \sup_{\dot{\varphi}} \left\{\dot{\varphi}\hat{q} - L^{\mathrm{f}}\left(\varphi, \dot{\varphi}; i_s\left(t\right)\right)\right\} \equiv H^{\mathrm{f}}\left(\hat{q} + q_s\left(t\right), \varphi\right)$$

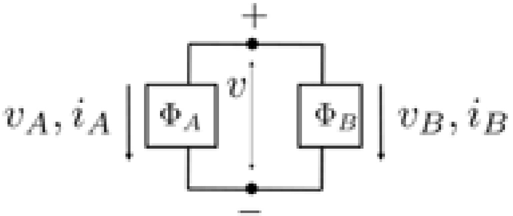
Matched Systems

- Charge-charge



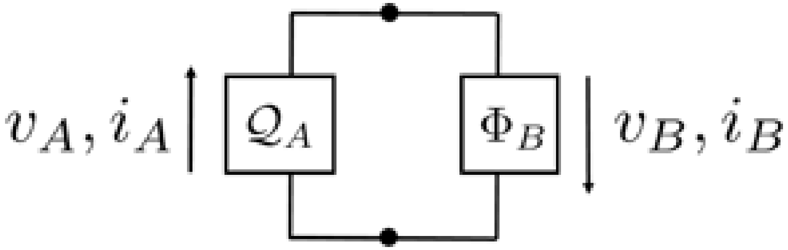
$$L^{\mathrm{c}}_{AB}(q,\dot{q}) = L^{\mathrm{c}}_{A}(q,\dot{q}) + L^{\mathrm{c}}_{B}(q,\dot{q}).$$

- Flux-flux



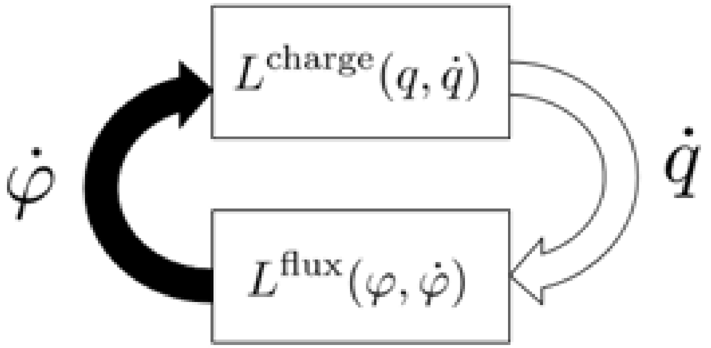
$$L^{\mathrm{f}}_{AB}(\varphi,\dot{\varphi}) = L^{\mathrm{f}}_{A}(\varphi,\dot{\varphi}) + L^{\mathrm{f}}_{B}(\varphi,\dot{\varphi}).$$

- Charge-flux



$$L\left(q_A,\dot{q}_A,\varphi_B,\dot{\varphi}_B\right) = L^{\mathrm{c}}_{A}(q_A,\dot{q}_A) + L^{\mathrm{f}}_{B}\left(\varphi_B,\dot{\varphi}_B\right) + (\lambda-1)q_A\dot{\varphi}_B + \lambda\varphi_B\dot{q}_A.$$

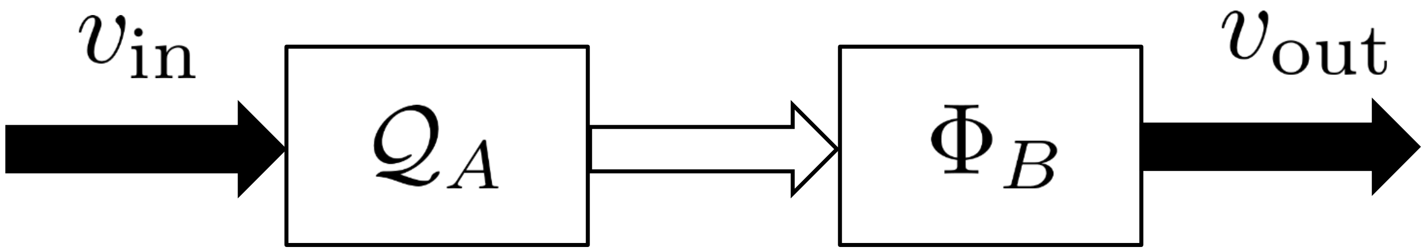
Charge-flux matching



$$
\begin{aligned}
H\left(q_A, \varphi_B, \hat{\varphi}_A, \hat{q}_B\right) &= \sup_{\dot{q}, \dot{\varphi}} \left\{ \dot{q}_A \hat{\varphi}_A + \dot{\varphi}_B \hat{q}_B - L\left(q_A, \dot{q}_A, \varphi_B, \dot{\varphi}_B\right) \right\} \\
&= H_A^{\mathrm{c}}\left(q_A, \hat{\varphi}_A - \lambda \varphi_B\right) + H_B^{\mathrm{f}}\left(\hat{q}_B - (\lambda - 1) q_A, \varphi_B\right)
\end{aligned}
$$

$$
\{f, g\} = \left(\frac{\partial f}{\partial q_A} \frac{\partial g}{\partial \hat{\varphi}_A} - \frac{\partial g}{\partial q_A} \frac{\partial f}{\partial \hat{\varphi}_A} \right) + \left(\frac{\partial f}{\partial \varphi_B} \frac{\partial g}{\partial \hat{q}_B} - \frac{\partial g}{\partial \varphi_B} \frac{\partial f}{\partial \hat{q}_B} \right).
$$

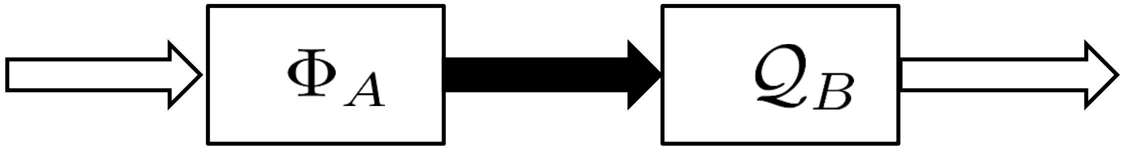
Cascading



$(\lambda = 1)$

$$H\left(q_A, \varphi_B, \hat{\varphi}_A, \hat{q}_B\right) \quad = H_A^{\mathrm{c}}\left(q_A, \hat{\varphi}_A - \varphi_B\right) + H_B^{\mathrm{f}}\left(\hat{q}_B, \varphi_B\right) - q_A v_{\mathrm{in}},$$

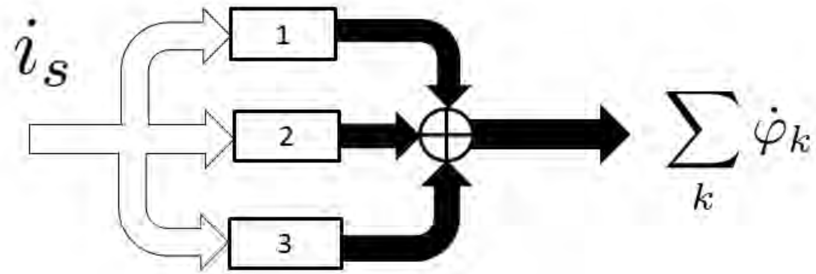
$$v_{\mathrm{out}} = \dot{\varphi}_B.$$

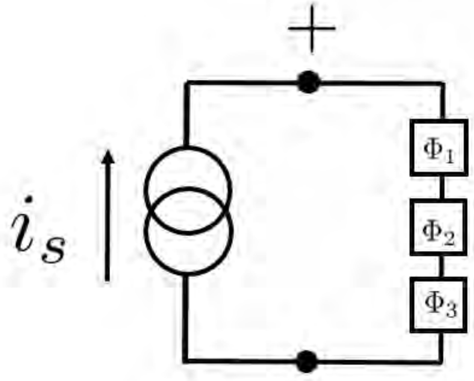


$A \leftrightarrow B,\ q \leftrightarrow \varphi,\ i \leftrightarrow v,\ (\lambda = 0).$

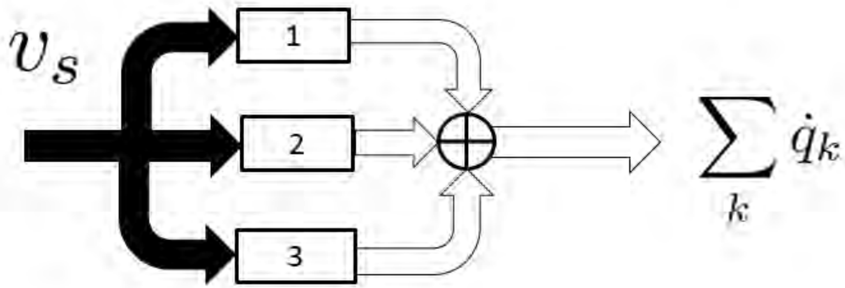
Networks

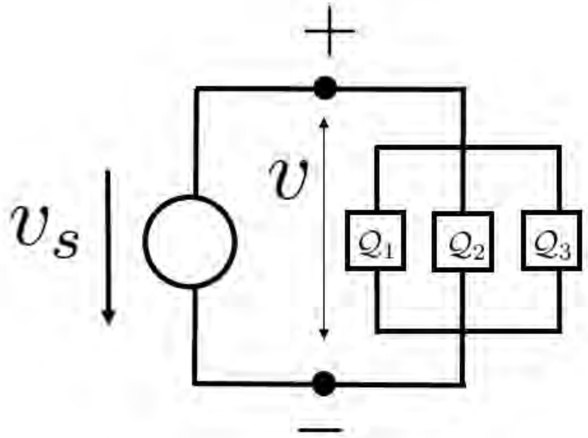
- Series





- Parallel





Resistors?

- Port-Hamiltonian networks (van der Schaft and Maschke)

  This is based on modifying the Poisson Bracket to include the dissipation terms

  But breaks the Jacobi identity!
- We need to use irreversible open systems in the quantum case
- Do we model using markovian models (*SLH*) or non-markovian?
- Passivity

$$\int_0^T vi\,dt \geq 0.$$

$$\text{Immittance } Z(s) \text{ LPR}$$

$$\mathrm{Re}Z(s) \geq 0 \text{ for } \mathrm{Re}s > 0,$$

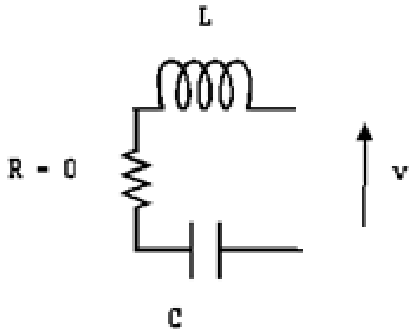
$$\mathrm{Re}Z(i\omega) \equiv 0.$$

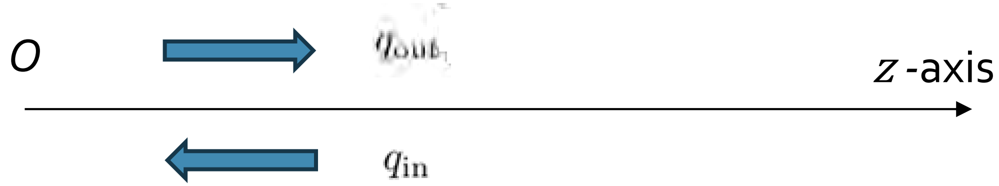
$$\text{Scattering } G(s) = \tfrac{1-Z(s)}{1+Z(s)} \text{ LBR}$$

$$G(s)^* G(s) \leq I \text{ for } \mathrm{Re}s > 0,$$

$$G(i\omega) \text{ unitary.}$$

Circuits + Transmission Lines





Circuits (LC Oscillator)

- Lagrangian: $L(q,\dot q) = \frac{1}{2}L\dot q^2 - \frac{1}{2C}q^2 + q \cdot v(t)$
- conjugate momentum: $p = \dfrac{\partial L}{\partial \dot q} = L\,\dot q$
- Hamiltonian: $H(q,p) = \dfrac{p^2}{2L} + \dfrac{q^2}{2C}$;
- EoM: $L\,\ddot q = -\dfrac{1}{C}q + v(t)$
- Energy: $\mathscr{E}_{\rm osc.} = \frac{1}{2}L\dot q^2 + \dfrac{q^2}{2C}$
- Power: $\dfrac{d}{dt}\mathscr{E}_{\rm osc.} = \dot q\, v(t)$

Transmission Line (TL)

- charge density: $Q(z,t)$
- current density: $I(z,t) = Q_{,t}$
- voltage density: $V(z,t) = -\dfrac{1}{\kappa}Q_{,z}$

$$\begin{aligned} I_{,t} + V_{,z} &= 0, \\ V_{,t} + I_{,z} &= 0. \end{aligned}$$

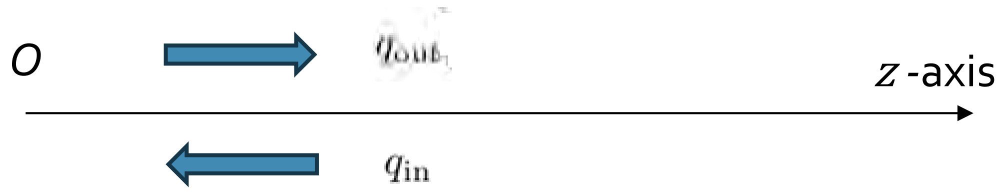
Telegrapher's equation

$$Q_{,tt} - \frac{1}{c^2}Q_{,zz} = 0$$

(wave speed) $c = \dfrac{1}{\sqrt{\ell\kappa}}$

$$Q(z,t) = q_{\rm in}\left(t + \frac{z}{c}\right) + q_{\rm out}\left(t - \frac{z}{c}\right)$$

Circuits + Transmission Lines

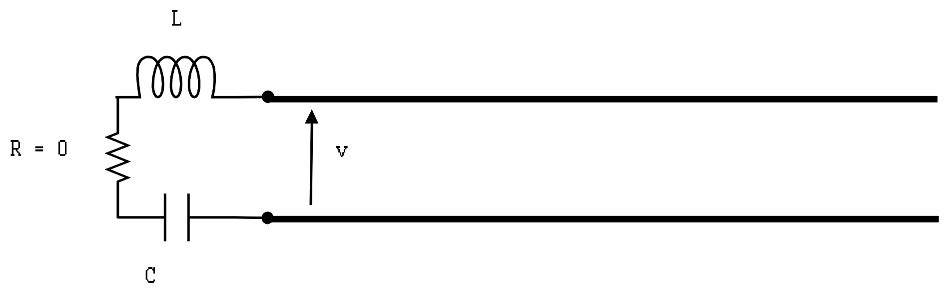


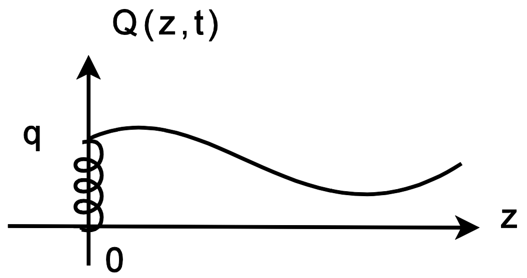
Transmission Line (TL)

- charge density: $Q(z,t)$
- current density: $I(z,t) = Q_{,t}$
- voltage density: $V(z,t) = -\frac{1}{\kappa} Q_{,z}$

- Lagrangian density: $\mathring{\mathcal{L}}(Q, Q_{,z}, Q_{,t}) = \frac{1}{2}\ell\, Q_{,t}^2 - \frac{1}{2\kappa} Q_{,z}^2$
- conjugate momentum: $\pi = \frac{\delta \mathring{\mathcal{L}}}{\partial Q_{,z}} = \ell\, Q_{,t}$
- Hamiltonian: $\mathring{\mathcal{H}} = \frac{\pi^2}{2\ell} + \frac{Q_{,z}^2}{2\kappa}$;
- Interior Energy: $\mathscr{E}_{\rm TL} = \int_0^\infty \mathring{\mathcal{H}}\, dz$
- Energy conservation: $\mathring{\mathcal{H}}_{,t} + \mathcal{S}_{,z} = 0$ with flux $\mathcal{S} = -\frac{1}{\kappa} Q_{,t} Q_{,z} \equiv I \cdot V$.
- Power: $\frac{d}{dt}\mathscr{E}_{\rm TL} = -I \cdot V|_{z=0}^{\infty} = -I(0,t) \cdot V(0,t)$.

Lamb Model (1900)





Lamb Model $q(t) \equiv Q(0,t)\ \dot{q}(t) \equiv I(0,t)$

Total Energy $\mathscr{E}_{\text{total}} = \mathscr{E}_{\text{osc.}} + \mathscr{E}_{\text{TL}}$

$$\frac{d}{dt}\mathscr{E}_{\text{total}} = \dot{q}\, v(t) + I(0,t)\, V(0,t)$$

$\mathscr{E}_{\text{total}}$ is conserved if $v(t) \equiv -V(0,t)$.

$$\begin{aligned} I(z,t) &= Q_{,t} = \dot{q}_{\text{in}}(t + \frac{z}{c}) + \dot{q}_{\text{out}}(t - \frac{z}{c}), \\ V(z,t) &= -\frac{1}{\kappa} Q_{,z} = -R\,\dot{q}_{\text{in}}(t + \frac{z}{c}) + R\dot{q}_{\text{out}}(t - \frac{z}{c}), \\ (R &= \frac{1}{\kappa c} = \sqrt{\frac{\ell}{\kappa}}) \end{aligned}$$

$$V(z,t) = R\,I(z,t) - 2\,R\,\dot{q}_{\text{in}}(t + \frac{z}{c})$$

$$v(t) = -R\,\dot{q} + F(t)$$

⇩

$$\boxed{L\,\ddot{q} + R\,\dot{q} + \frac{1}{C} q = F(t).}$$

Finite Transmission Lines

$$\mathcal{L}_{[a,b]} = \mathring{\mathcal{L}}(Q, Q_{,z}, Q_{,t})\, 1_{[a,b]} + G(Q, Q_{,t})\, [\delta_a(z) + \delta_b(z)].$$

Note:

1. $\mathcal{L}_{[a,b]} + \mathcal{L}_{[b,c]} = \mathcal{L}_{[a,c]}$
2. $\dfrac{\delta G}{\delta Q_{,z}} \equiv 0$
3. $\dfrac{\partial}{\partial z} 1_{[a,b]} = \delta_a - \delta_b$

The Euler-Lagrange equations

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\Big(\frac{\delta \mathcal{L}_{[a,b]}}{\delta Q_{,t}}\Big) + \frac{\partial}{\partial z}\Big(\frac{\delta \mathcal{L}_{[a,b]}}{\delta Q_{,z}}\Big) - \frac{\delta \mathcal{L}_{[a,b]}}{\delta Q} \\
&= \left\{\frac{\partial}{\partial t}\Big(\frac{\delta \mathring{\mathcal{L}}}{\delta Q_{,t}}\Big) + \frac{\partial}{\partial z}\Big(\frac{\delta \mathring{\mathcal{L}}}{\delta Q_{,z}}\Big) - \frac{\delta \mathring{\mathcal{L}}}{\delta Q}\right\} 1_{[a,b]} \\
&\quad + \left\{\frac{\partial}{\partial t}\Big(\frac{\delta G}{\delta Q_{,t}}\Big) - \Big(\frac{\delta G}{\delta Q}\Big) + \frac{\delta \mathring{\mathcal{L}}}{\delta Q_{,z}}\right\} \delta_a \\
&\quad + \left\{\frac{\partial}{\partial t}\Big(\frac{\delta G}{\delta Q_{,t}}\Big) - \Big(\frac{\delta G}{\delta Q}\Big) - \frac{\delta \mathring{\mathcal{L}}}{\delta Q_{,z}}\right\} \delta_b.
\end{aligned}$$

$\Downarrow$

Boundary conditions

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(\frac{\delta G}{\delta Q_{,t}}\Big) - \Big(\frac{\delta G}{\delta Q}\Big) + \frac{\delta \mathcal{L}}{\delta Q_{,z}} &= 0, \quad \text{at } z = a; \\
\frac{\partial}{\partial t}\Big(\frac{\delta G}{\delta Q_{,t}}\Big) - \Big(\frac{\delta G}{\delta Q}\Big) - \frac{\delta \mathring{\mathcal{L}}}{\delta Q_{,z}} &= 0, \quad \text{at } z = b.
\end{aligned}$$

LC Oscillators Talking via a Finite T.L.

Boundary conditions

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\delta G}{\delta Q_{,t}}\right) - \left(\frac{\delta G}{\delta Q}\right) + \frac{\delta \mathring{\mathcal{L}}}{\delta Q_{,z}} &= 0, \quad \text{at } z = a; \\ \frac{\partial}{\partial t}\left(\frac{\delta G}{\delta Q_{,t}}\right) - \left(\frac{\delta G}{\delta Q}\right) - \frac{\delta \mathring{\mathcal{L}}}{\delta Q_{,z}} &= 0, \quad \text{at } z = b. \end{aligned}$$

$$G = \frac{1}{2} L(z)\, Q_{,t}^2 - \frac{1}{2C(z)}\, Q^2 + e_z(t) \cdot Q.$$

$$\begin{aligned} \mathcal{H}_{[a,b]} &= \pi \cdot Q_{,t} + p_a \cdot \dot{q}_a\, \delta_a + p_b \cdot \dot{q}_b \delta_b - \mathcal{L}_{[a,b]} \\ &= \mathring{\mathcal{H}}(Q, Q_{,z}, Q_{,t})\, 1_{[a,b]} + H(Q, Q_{,t})\, [\delta_a(z) + \delta_b(z)]. \end{aligned}$$

Basic idea (semi-infinite, $b = \infty$) Yurke Denker (1984)

See also Jeltsema, v.d. Schaft, Rep. Math. Phys. (2009)

$$\begin{aligned} H &= \frac{\delta G}{\delta Q_{,t}} Q_{,t} - G \\ &= \frac{1}{2} L(z)\, Q_{,t}^2 + \frac{1}{2C(z)} Q^2. \end{aligned}$$

$$\frac{d}{dt}\mathscr{E}_{[a,b]} = -S|_a^b - I_a \cdot V_a + I_b \cdot V_b = 0.$$

Models with Memory

$$\mathbf{L}\ddot{\mathbf{q}}(t) + \int_{-\infty}^{t} \boldsymbol{\Gamma}(t-t')\,\dot{\mathbf{q}}(t')dt' + \mathbf{K}\mathbf{q}(t) = \mathbf{F}(t),$$

frequency dependent resistance

$$\mathbf{R}(\omega) \triangleq \mathscr{L}\boldsymbol{\Gamma}\left[0^+ - i\omega\right].$$

$$\mathbf{S}[s] \triangleq \frac{\mathbf{L}s^2 - s\,\mathscr{L}\boldsymbol{\Gamma}[s] + \mathbf{K}}{\mathbf{L}s^2 + s\,\mathscr{L}\boldsymbol{\Gamma}[s] + \mathbf{K}}, \quad \mathrm{Re}s > 0,$$

Spectral Density

$$\boldsymbol{\Gamma}(t) = \frac{1}{\pi}\int_0^{\infty} \frac{\mathbf{J}(\omega)}{\omega} \cos\omega t\, d\omega\, \theta(t).$$

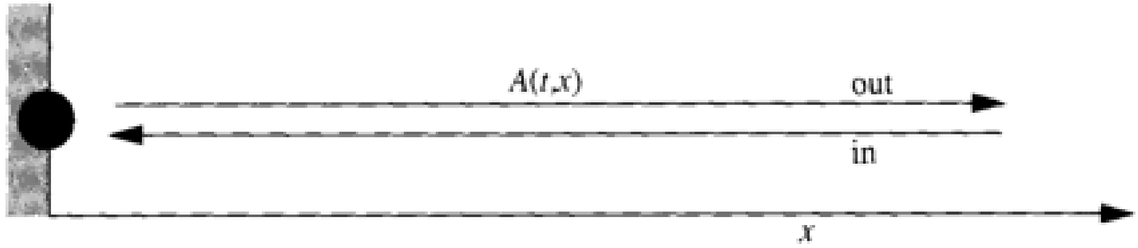
$$\mathbf{J}(\omega) \triangleq \omega\,\mathrm{Re}\mathbf{R}(\omega).$$

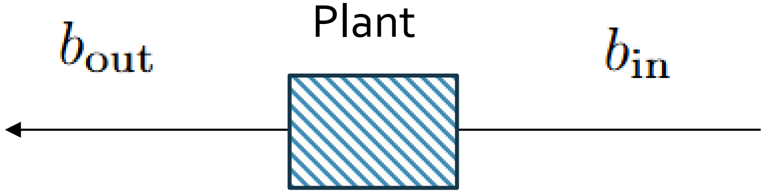
Transfer function is linear bounded real iff $\mathscr{L}\boldsymbol{\Gamma}[s]$ (**immittance)** is linear positive real.

JG, Journ. Math. Phys., 57, 122101 (2016)

Markov Limits

Gardiner-Collett



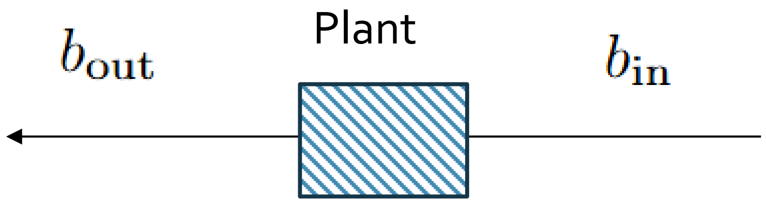


Quantize the plant – quantum mechanical system

Quantize the wire – boson field

$$\boxed{[b_{\text{in}}(t_1), b_{\text{in}}(t_2)^*] = \delta(t_1 - t_2)}$$

SLH Theory



Unitary Dynamics (Wick order)

$$\frac{d}{dt}U_t = b_{\text{in}}^*(S-1)U_t b_{\text{in}} + b_{\text{in}}^* L U_t - L^* S U_t b_{\text{in}} - \tfrac{1}{2}L^*L + iH)U_t$$

S - unitary, L bounded, $H = H^*$

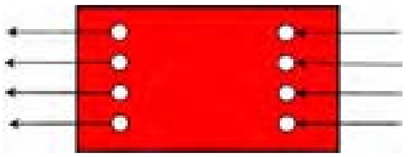
$$\begin{aligned} \text{(Heisenberg) } j_t(X) &= U_t^*[X \otimes I]U_t \\ \text{(output) } \int_0^t b_{\text{out}}(\tau)\,d\tau &= U_t^*[I \otimes \int_0^t b_{\text{in}}(\tau)\,d\tau]U_t \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}j_t(X) &= b_{\text{in}}^*(S^*XS - X)U_t b_{\text{in}} + b_{\text{in}}^*(S^*[X,L]I - j_t([L^*,X]S)b_{\text{in}} \\ &\quad + j_t(\frac{1}{2}[L^*,X]L + \frac{1}{2}L^*[X,L] - i[X,H]), \\ b_{\text{out}}(t) &= j_t(S)b_{\text{in}} + j_t(L). \end{aligned}$$

SLH Formalism (Markovian)

- Quantum white noise $[b_j(t), b_k^*(s)] = \delta_{jk}\,\delta(t-s)$
- Hamiltonian *H* $H^* = H$
- Coupling/Collapse Operators *L*

$$L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}$$

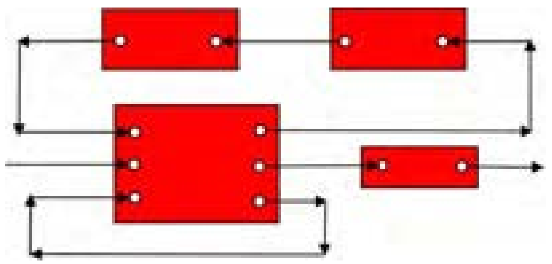


- Scattering Operator *S*

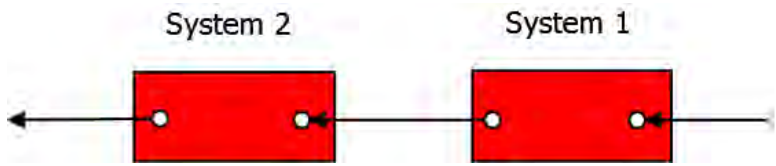
$$S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix}, \qquad S^{-1} = S^*$$

Quantum Networks (Cavity QED)

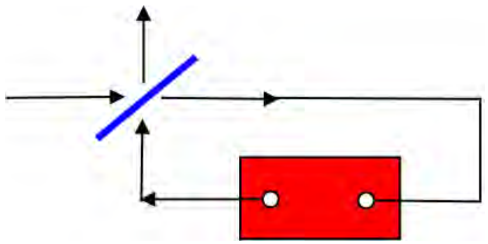
How to connect models?



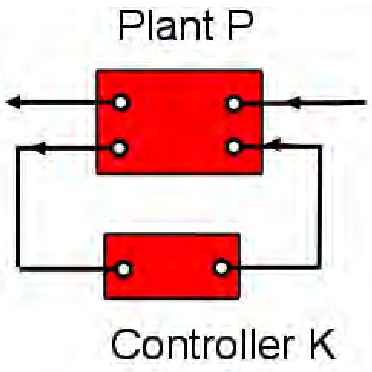
Cascaded models



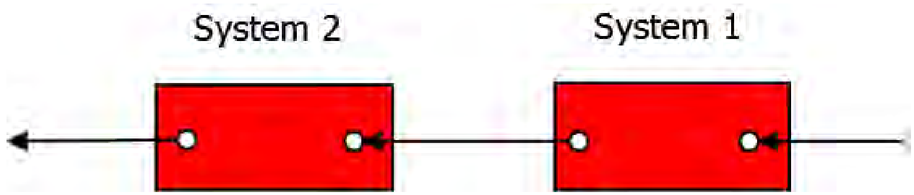
Algebraic loops



Feedback Control



The Series Product

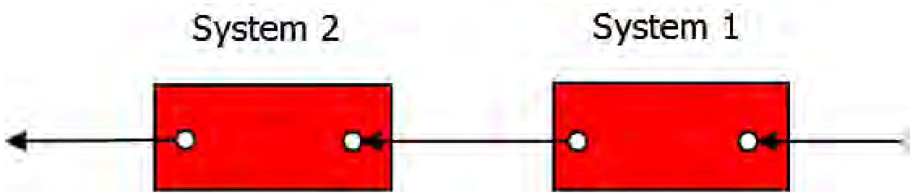


The cascaded system in the **instantaneous feedforward** limit is equivalent to the single component

$$(S_2, L_2, H_2) \triangleleft (S_1, L_1, H_1) = \left(S_2 S_1, L_2 + S_2 L_1, H_1 + H_2 + \mathrm{Im}\left\{L_2^\dagger S_2 L_1\right\}\right).$$

J. G., M.R. James, *The Series Product and Its Application to Quantum Feedforward and Feedback Networks* IEEE Transactions on Automatic Control, 2009.

The Series Product



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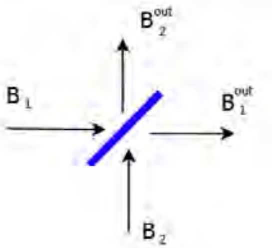
J. G., M.R. James, *The Series Product and Its Application to Quantum Feedforward and Feedback Networks* IEEE Transactions on Automatic Control, 2009.

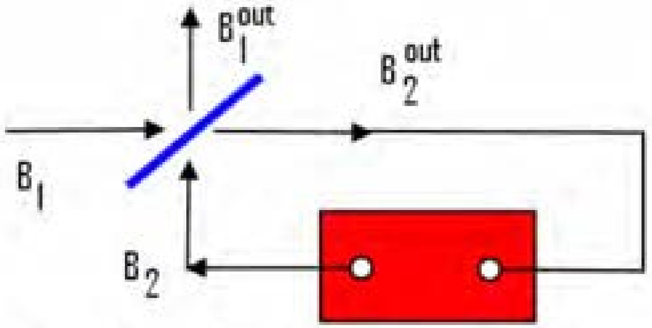
Beam-splitters

$$\begin{bmatrix} B_1^{\text{out}} \\ B_2^{\text{out}} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$



beamsplitter $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$,

and *in-loop component* $(S_0, L_0, 0)$:



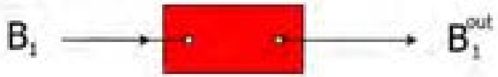
$$dB_2 = S_0 dB_2^{\text{out}} + L_0 dt = S_0(S_{21} dB_1 + S_{22} dB_2) + L_0 dt$$

$$\Rightarrow dB_1^{\text{out}} = S_{11} dB_1 + S_{12} dB_2 \equiv \hat{S}_0 dB_1 + \hat{L}_0 dt$$

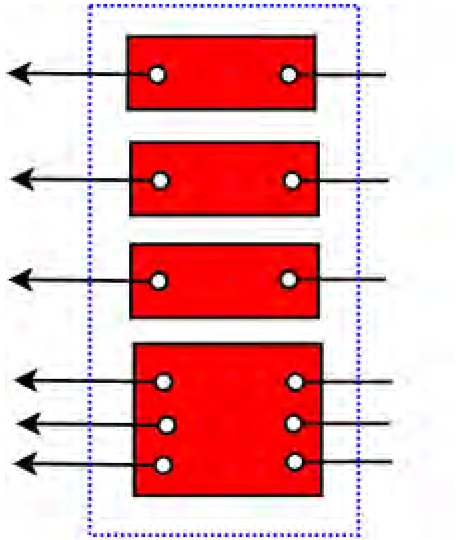
where

$$\hat{S}_0 = S_{11} + S_{12}(I - S_0 S_{22})^{-1} S_0 S_{21}, \quad \hat{L}_0 = S_{12}(I - S_{22})^{-1} S_0 L_0.$$

Equivalent component $(\hat{S}_0, \hat{L}_0, \hat{H}_0)$:



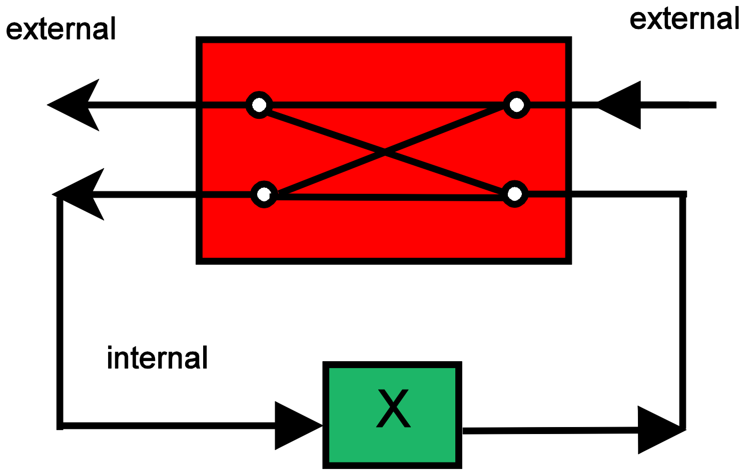
Network Rule # 1
Open loop systems in parallel



Models $(S_j, L_j, H_j)_{j=1}^{n}$ in parallel

$$\left(\begin{bmatrix} S_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & S_n \end{bmatrix}, \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}, H_1 + \cdots + H_n \right).$$

Network Rule # 2
Feedback Reduction Formula



$$S = \begin{bmatrix} \mathsf{S}_{\mathtt{ii}} & \mathsf{S}_{\mathtt{ie}} \\ \mathsf{S}_{\mathtt{ei}} & \mathsf{S}_{\mathtt{ee}} \end{bmatrix}, \ \mathsf{L} = \begin{bmatrix} \mathsf{L}_{\mathtt{i}} \\ \mathsf{L}_{\mathtt{e}} \end{bmatrix}$$

The reduced model obtained by eliminating all the internal channels (instantaneous feedback) is determined by the operators $(\mathsf{S}^{\mathrm{fb}}, \mathsf{L}^{\mathrm{fb}}, \mathsf{H}^{\mathrm{fb}})$ given by

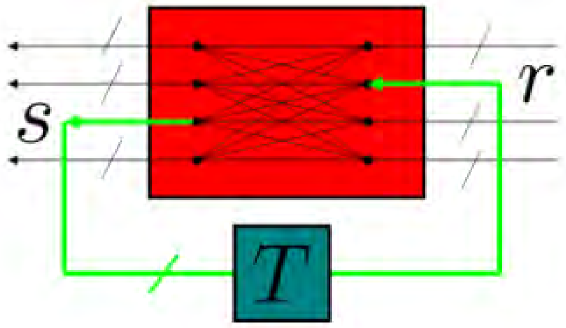
$$\begin{aligned} \mathsf{S}^{\mathrm{fb}} &= \mathsf{S}_{\mathtt{ee}} + \mathsf{S}_{\mathtt{ei}} X \left(1 - \mathsf{S}_{\mathtt{ii}} X\right)^{-1} \mathsf{S}_{\mathtt{ie}}, \\ \mathsf{L}^{\mathrm{fb}} &= \mathsf{L}_{\mathtt{e}} + \mathsf{S}_{\mathtt{ei}} X \left(1 - \mathsf{S}_{\mathtt{ii}} X\right)^{-1} \mathsf{L}_{\mathtt{i}}, \\ \mathsf{H}^{\mathrm{fb}} &= \mathsf{H} + \sum_{i=\mathtt{i},\mathtt{e}} \mathrm{Im} \mathsf{L}_j^{\dagger} X \mathsf{S}_{j\mathtt{i}} \left(1 - \mathsf{S}_{\mathtt{ii}} X\right)^{-1} \mathsf{L}_{\mathtt{i}}. \end{aligned}$$

J. G., M.R. James, *Quantum Feedback Networks: Hamiltonian Formulation*, Commun. Math. Phys., 1109-1132, Volume 287, Number 3 / May, 2009.

The rules are very similar to classical linear systems

The quantum **model matrix** for $G \sim (S, L, H)$:

$$\mathsf{V} = \begin{bmatrix} -\frac{1}{2}L^*L - iH & -L^*S \\ L & S \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sum_j L_j^* L_j - iH & -\sum_j L_j^* S_{j1} & \cdots & -\sum_j L_j^* S_{jm} \\ L_1 & S_{11} & \cdots & S_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ L_n & S_{n1} & \cdots & S_{nn} \end{bmatrix} = \begin{bmatrix} \mathsf{V}_{00} & \mathsf{V}_{01} & \cdots & \mathsf{V}_{0m} \\ \mathsf{V}_{10} & \mathsf{V}_{11} & \cdots & \mathsf{V}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{V}_{n0} & \mathsf{V}_{n1} & \cdots & \mathsf{V}_{nn} \end{bmatrix}.$$



The feedback reduction yields the model matrix

$$\left[\mathscr{F}_{(r,s)}(\mathsf{V}, T)\right]_{\alpha\beta} = \mathsf{V}_{\alpha\beta} + \mathsf{V}_{\alpha r} T \left(1 - \mathsf{V}_{rs} T\right)^{-1} \mathsf{V}_{s\beta}$$

for $\alpha \neq r$ and $\beta \neq s$.

Hamiltonian Formulation of SLH

$$U_t = e^{+iH_0 t} e^{-iHt}$$

H_0 is the Stone generator for the shift along the wires

Chebotarev-Gregoratti "Network" Hamiltonian

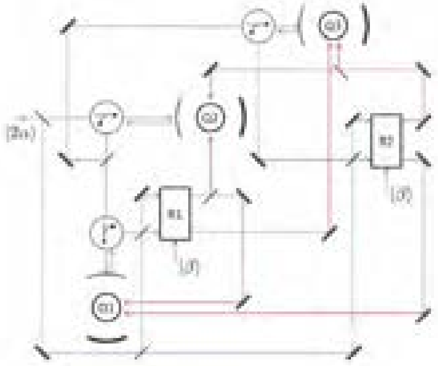
$$-iH\Psi = -i\text{“}H_0\Psi\text{''} - \sum_v \left(\tfrac{1}{2} L_v^* L_v + iH\right)\Psi - \sum_v L_v^* S_v\, b_v(+)\Psi,$$

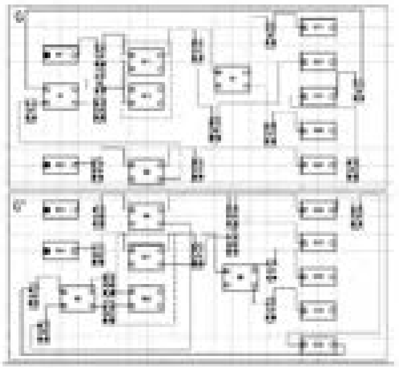
$$b_v(-)\Psi = L_v \Psi + S_v b_v(+)\Psi$$

$$\mathrm{V} = \begin{bmatrix} -\frac{1}{2}L^*L - iH & -L^*S \\ L & S \end{bmatrix}$$

Quantum Fly-ball Governor

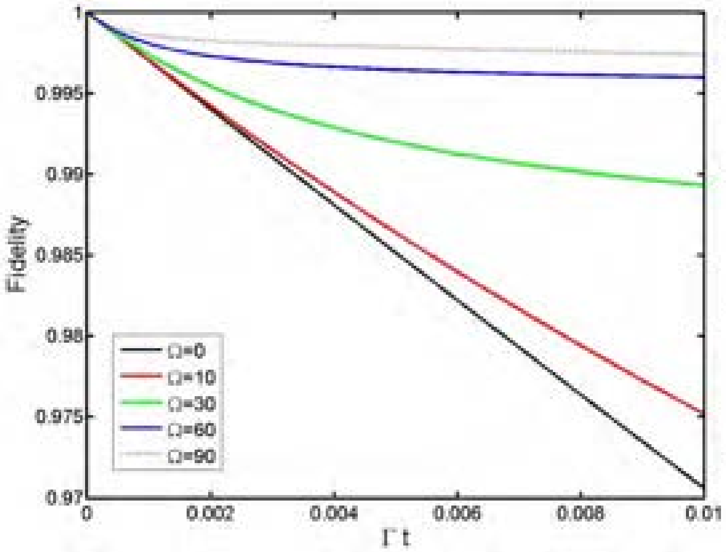
Set up in QHDL





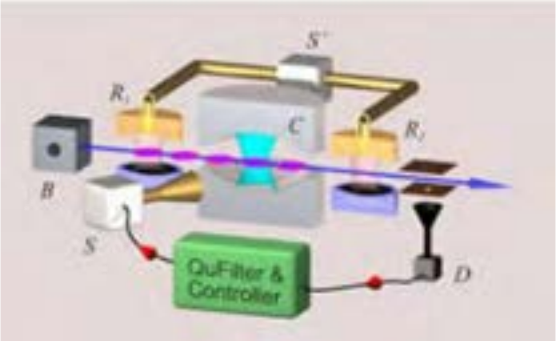
Network rules yield the overall "*SLH*"
From which we deduce the master equation

$$\begin{aligned}\dot{\rho}_t &= -i[H,\rho_t] + \sum_{i=1}^{7}\left(L_i\rho_t L_i^* - \frac{1}{2}\{L_i^* L_i, \rho_t\}\right)\\ H &= \sqrt{2}\Omega\Pi_g^{(R_1)}\Pi_h^{(R_2)}X_1 + \sqrt{2}\Omega\Pi_h^{(R_1)}\Pi_g^{(R_2)}X_3 - \Omega\Pi_g^{(R_1)}\Pi_g^{(R_2)}X_2\end{aligned}$$

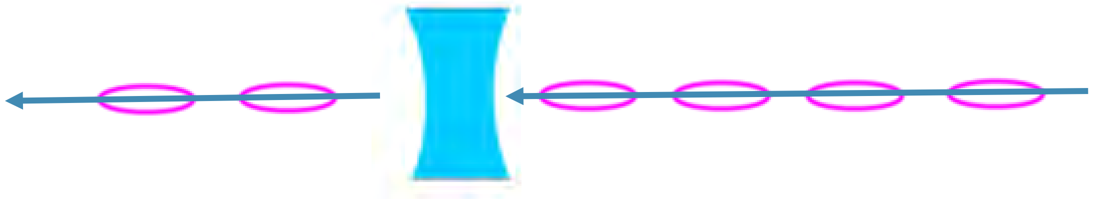


$$\begin{aligned}L_1 &= \frac{\alpha}{\sqrt{2}}\{\sigma_{hg}^{(R_1)}(1+Z_1Z_2)\\ &\quad +\Pi_h^{(R_1)}(1-Z_1Z_2)\}\\ L_2 &= \frac{\alpha}{\sqrt{2}}\{\sigma_{gh}^{(R_1)}(1-Z_1Z_2)\\ &\quad +\Pi_g^{(R_1)}(1+Z_1Z_2)\}\\ L_3 &= \frac{\alpha}{\sqrt{2}}\{\sigma_{hg}^{(R_2)}(1+Z_3Z_2)\\ &\quad +\Pi_h^{(R_2)}(1-Z_3Z_2)\}\\ L_4 &= \frac{\alpha}{\sqrt{2}}\{\sigma_{gh}^{(R_2)}(1-Z_3Z_2)\\ &\quad +\Pi_g^{(R_2)}(1+Z_3Z_2)\}\end{aligned}$$

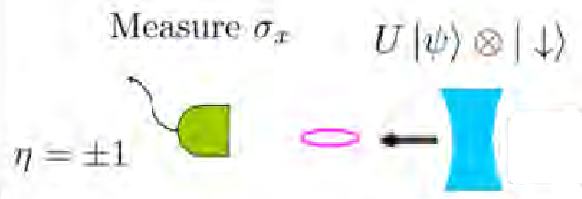
Discrete Approximations



Cavity



$$
\begin{aligned}
U &= \exp\{\sqrt{\tau} L \otimes \sigma^* - \sqrt{\tau} L^* \otimes \sigma - i\tau H \otimes I_2\} \\
&\simeq 1 + \sqrt{\tau} L \otimes \sigma^* - \sqrt{\tau} L^* \otimes \sigma - \tau(\frac{1}{2} L^* L + iH) \otimes I_2 + \cdots .
\end{aligned}
$$



$$p_{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{\tau}\langle\psi|L + L^*|\psi\rangle + \cdots$$

$$|\psi_\eta\rangle \propto |\psi\rangle + \sqrt{\tau} L|\psi\rangle\,\eta - \tau(\frac{1}{2} L^* L + iH)|\psi\rangle + \cdots .$$

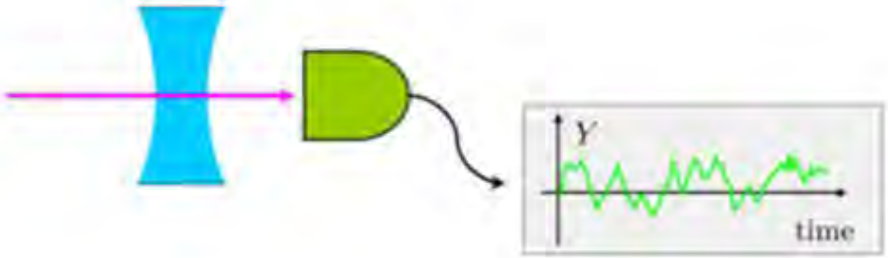
$$\frac{1}{\sqrt{\tau}} \sum_{j=1}^{\lfloor t/\tau \rfloor} \sigma_j \to \int_0^t b_{\text{in}}(s)\, ds$$

We may take a continuum limit we have (central limit effect)

$$\tau \hookrightarrow dt \quad \sqrt{\tau}\eta \hookrightarrow dY$$

The limit equation is (quantum Zakai equation)

$$d|\chi_t\rangle = L|\chi_t\rangle\, dY_t - (\frac{1}{2}L^* L + iH)|\chi_t\rangle\, dt.$$



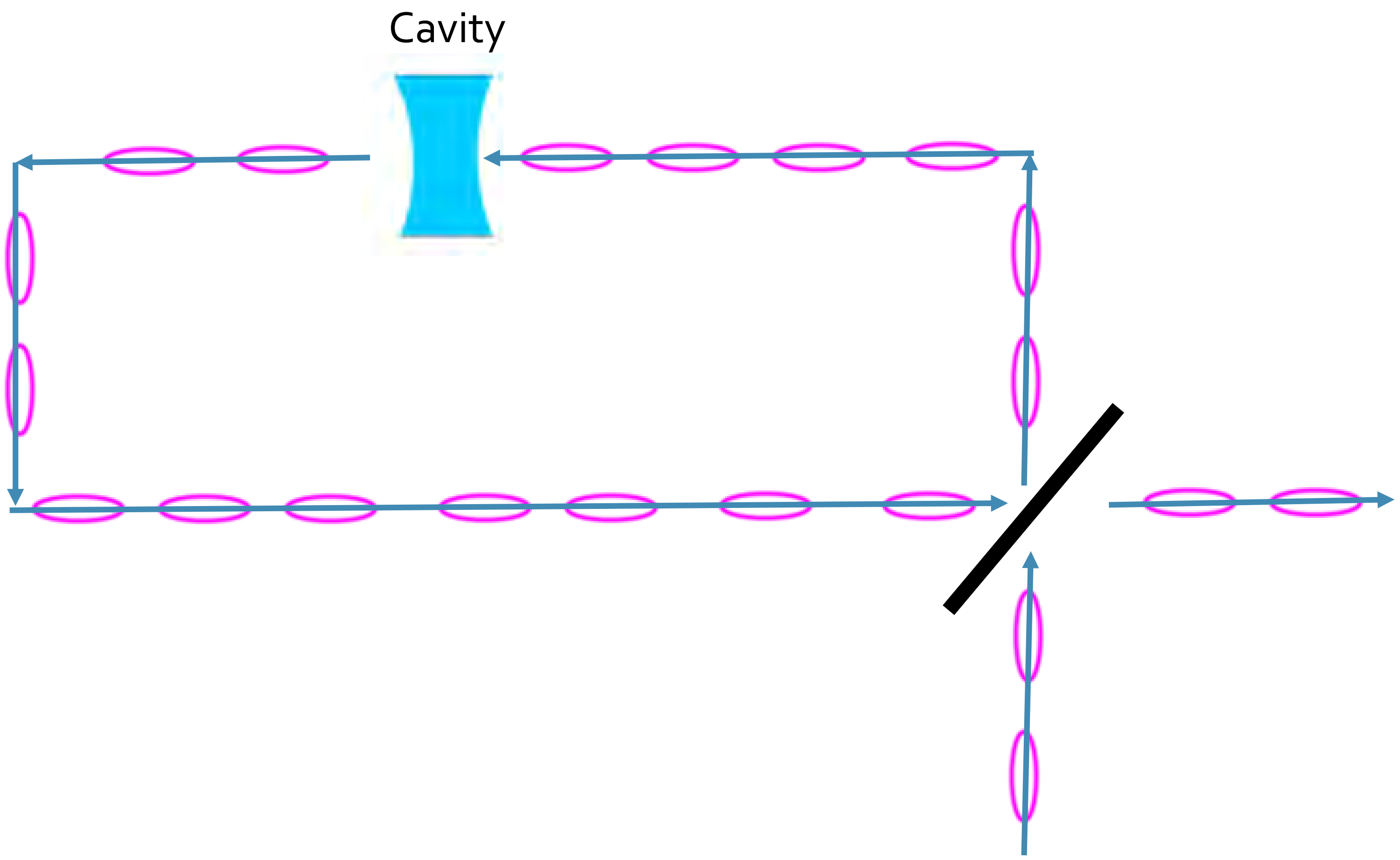
For the normalized state $|\psi_t\rangle = |\chi_t\rangle / \, \|\chi_t\|$ we find (**Stochastic Schrödinger equation**)

$$d|\psi_t\rangle = -\, iH|\psi_t\rangle\, dt - \frac{1}{2}(L - \lambda_t)^*(L - \lambda_t)|\psi_t\rangle\, dt + (L - \lambda_t)|\psi_t\rangle\, dI_t.$$

$$dI_t \quad = \quad dY_t - \lambda_t\, dt.$$

(**Innovations** – Wiener process)

Discrete Approximations



NON-MARKOVIAN QUANTUM MODELS

Heat Bath with memory

Papers

- *Quantum feedback control of a two-level atom network closed by a semi-infinite waveguide*

  H. Ding, G. Zhang, Mu-Tian Cheng, Guoqing Cai arXiv:2306.06373v2

- *Quantum coherent feedback control of an N-level atom with multiple excitations*

  H. Ding, G. Zhang IEEE TAC, (2024)

- *Quantum coherent and measurement feedback control based on atoms coupled with a semi-infinite waveguide*

  H. Ding, N.H. Amini, G. Zhang, JG SIAM Journal Control & Opt. (0), S231-S257 (2025)

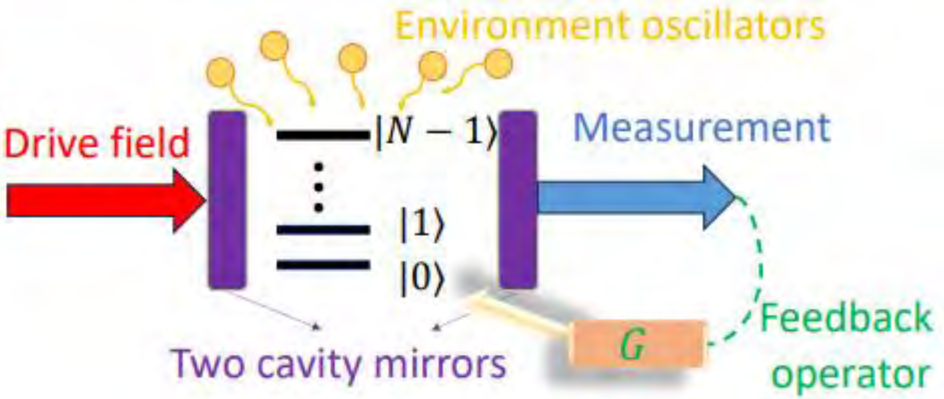
- *On non-Markovian quantum control dynamics*

  H. Ding, N.H. Amini, G. Zhang, JG *(to appear IEEE TAC)*

- *Reproducing Kernel Hilbert Space Approach to Non-Markovian Quantum Stochastic Models*

  JG, H. Ding, N.H. Amini Journ. Math. Phys, 66 042102 (2025)

NON-MARKOVIAN QUANTUM CONTROL FOR OPEN SYSTEMS



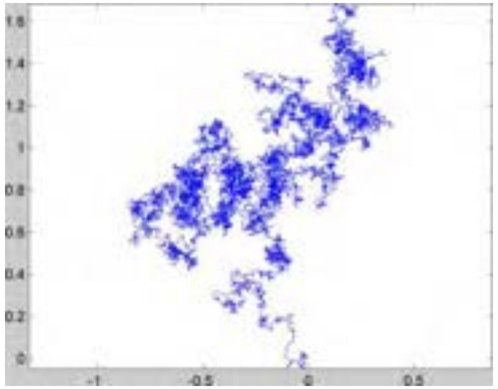
$$H = \omega_c a^\dagger a + \sum_\omega \omega b_\omega^\dagger b_\omega + \sum_{n=0}^{N-1} \omega_n |n\rangle\langle n| + H_I,$$

$$\tilde{H}_I = \sum_{n=1}^{N-1} g_n \left(e^{-i\Delta_n t}\sigma_n^- a^\dagger + e^{i\Delta_n t}\sigma_n^+ a\right) + \sum_{n=1}^{N-1}\sum_\omega \chi_\omega^{(n)} \left(\sigma_n^- b_\omega^\dagger + \sigma_n^+ b_\omega\right),$$

Non-Markovian bath

dissipator $L = \sum_{n=1}^{N-1} L_n = \sum_{n=1}^{N-1} \sqrt{\kappa_n}|n-1\rangle\langle n|,$ kernel $\frac{\gamma}{2}e^{-\gamma|t-s|-i\Omega(t-s)}$

Quantum State Diffusion (Non-Markovian)



- Postulated by W. Strunz and L. Diósi
- Complex (classical) Gaussian stochastic process: $\boxed{z_t \equiv \dot{\xi}_t}$

$$\mathbb{E}[z_t z_s] = 0, \quad \mathbb{E}[z_t^* z_s] = K(t,s) = K(s,t)^*.$$

- Stochastic equation with memory (**causal?**)

$$\frac{d}{dt}|\Psi_t\rangle = -iH|\Psi_t\rangle + L|\Psi_t\rangle z(t) - L^* \int_0^t K(t,s) \frac{\delta}{\delta z_s} |\Psi_t\rangle \, ds.$$

Microscopic Model - Bath

$$\mathfrak{H}_B = \Gamma(\mathfrak{f}_B)$$

e.g., $\mathfrak{f}_B = L^2(\mathbb{R}_+, d\omega)$.

- Creation/annihilation operators

$$[\hat{a}_\omega, \hat{a}^*_{\omega'}] = \delta(\omega - \omega').$$

- smeared fields

$$\hat{a}(g)^* = \int_0^\infty g(\omega)\hat{a}^*_\omega \, d\omega, \qquad \hat{a}(g) = \int_0^\infty g(\omega)^* \hat{a}_\omega \, d\omega$$

- The bath Hamiltonian is just oscillatory

$$e^{it\hat{H}_B}\hat{a}(g)e^{-it\hat{H}_B} \equiv \hat{a}(g_t), \text{ where } |g_t\rangle = e^{it\hat{h}_B}|g\rangle.$$

- The bath correlation function (**kernel**) is

$$\boxed{K(t,s) \triangleq \langle g_t | g_s \rangle = \langle g | e^{-i(t-s)\hat{h}_B} g \rangle.}$$

- For a given vector $|f\rangle \in \mathfrak{f}_B$, its *exponential vector* is defined as

$$|e^f\rangle_B = e^{\hat{a}(f)^*}|\mathrm{vac}\rangle_B \equiv |\mathrm{vac}\rangle \oplus |f\rangle \oplus \left(\frac{1}{\sqrt{2!}}|f\rangle \otimes |f\rangle\right) \oplus \cdots .$$

- The exponential vectors form a total subset in $\mathfrak{H}_B$ and we note that

$$\langle e^f|e^g\rangle_B = e^{\langle f|g\rangle}.$$

- The **Complex Wave representation** of a vector $|\Psi\rangle \in \mathfrak{H}_B$ is given by

$$\boxed{\tilde{\Psi}(f) = \langle e^f|\Psi\rangle.}$$

- We also have the resolution of identity

$$\boxed{\int_{\mathfrak{f}_B} |e^f\rangle\langle e^f|\, \tilde{\mathbb{P}}[df] = \hat{I}_B.}$$

$$\hat{H}_{\text{Tot.}} = \hat{H} \otimes \hat{I}_B + \hat{I}_S \otimes \hat{H}_B + i\hat{L} \otimes \hat{a}(g)^* - i\hat{L}^* \otimes \hat{a}(g).$$

- Move into the interaction picture with respect to the bath

$$\frac{d}{dt}\hat{U}_t = -i\hat{\Upsilon}_t \hat{U}_t, \qquad -i\hat{\Upsilon}_t = -i\hat{H} \otimes \hat{I}_B + \hat{L} \otimes \hat{Z}(t) - \hat{L}^* \otimes \hat{Z}(t)^*.$$

- Here we introduce the "***bath processes***"

$$\hat{Z}(t) = \hat{a}(g_t)^*.$$

- The commutation relations are

$$[\hat{Z}(t)^*, \hat{Z}(s)] = K(t,s)\, \hat{I}_B.$$

Input-System-Output Model

- Input-output relation

$$\hat{Z}_{\mathrm{out}}(t)^* = \hat{Z}_{\mathrm{in}}(t)^* + \int_0^t K(t,\tau) j_\tau(\hat{L}) d\tau.$$

- "Ehrenfest Equation" with memory!!!

$$\frac{d}{dt}\langle j_t(\hat{X})\rangle = \langle j_t(\frac{1}{i}[\hat{X},\hat{H}])\rangle + \int_0^t d\tau\, K(t,\tau)\langle j_t([\hat{L}^*,\hat{X}]) j_\tau(\hat{L})\rangle + \int_0^t d\tau\, K(t,\tau)^* \langle j_\tau(\hat{L}^*) j_t([\hat{X},\hat{L}])\rangle.$$

For each $f \in \mathfrak{f}$, we define its associated **complex trajectory** over the time interval $\mathbb{T}$ to be the function $\zeta(f) : \mathbb{T} \mapsto \mathbb{C} : t \mapsto \zeta_t(f)$ where

$$\boxed{\zeta_t(f) = \langle f | g_t \rangle_{\mathfrak{f}}.}$$

The space of complex trajectories will be denoted as $\mathscr{C}_K(\mathbb{T}, dt)$.

$$\boxed{\left(\hat{Z}_t^* - \zeta_t(f)^* \right) |e^f\rangle_B = 0.}$$

$$\boxed{K(t,s) \triangleq \langle g_t | g_s \rangle = \langle g | e^{-i(t-s)\hat{h}_B} g \rangle.}$$

$$\langle \Phi | \Psi \rangle = \int_{\mathfrak{f}_B} \langle \tilde{\Phi}(f) | \tilde{\Psi}(f) \rangle_S \, \tilde{\mathbb{P}}\left[df\right] = \int_{\mathscr{C}_K(\mathbb{T}, dt)} \langle \Phi(\zeta) | \Psi(\zeta) \rangle_S \, \mathbb{P}\left[d\zeta\right].$$

- In the Bargmann representation, $\hat{Z}(t)$ corresponds to multiplication by $\zeta_t(\cdot)$:

$$\hat{Z}(t)\,\tilde{\Psi}(f) = \zeta_t(f)\,\tilde{\Psi}(f).$$

- With the change of variable, $\hat{Z}(t)$ is multiplication by $\zeta_t(\cdot)$!

- For the case where $\mathbb{T} = \mathbb{R}$, the adjoint operator $\hat{Z}(t)^*$ is

$$\hat{Z}(t)^* \equiv \int_{-\infty}^{\infty} d\tau\, K\left(t, \tau\right) \frac{\delta}{\delta \zeta_\tau}.$$

Complex Trajectory Unravellings

- **Causality?**

Let $|\Psi_t(\zeta)\rangle_S$ be the hybrid complex wave representation of $|\Psi_t\rangle = \hat{U}_t\,|\phi \otimes \mathrm{vac}\rangle$, then

$$\frac{\delta}{\delta \zeta_\tau} |\Psi_t(\zeta)\rangle_S = 0, \quad \text{whenever} \quad \tau \notin [0, t].$$

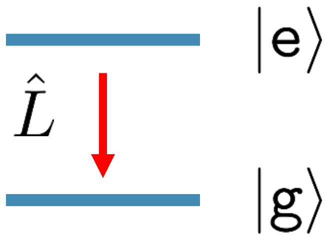
- **Microscopic Derivation**

An unravelling is given on the space $\mathscr{C}_K(\mathbb{T}, dt)$ of complex trajectories with measure $\mathbb{P}$. The unravelling $\zeta \mapsto |\Psi_t(\zeta)\rangle_S$ satisfies the Diósi-Strunz equation with initial condition $|\Psi_0(\zeta)\rangle_S = |\phi\rangle_S$ for all ζ.

Exactly Solvable Model

- Jaynes-Cummings model: 2-level atom coupled to a bosonic bath

$$\hat{H} = \omega_0 |\mathsf{e}\rangle\langle\mathsf{e}| \text{ and } \hat{L} = |\mathsf{g}\rangle\langle\mathsf{e}|.$$



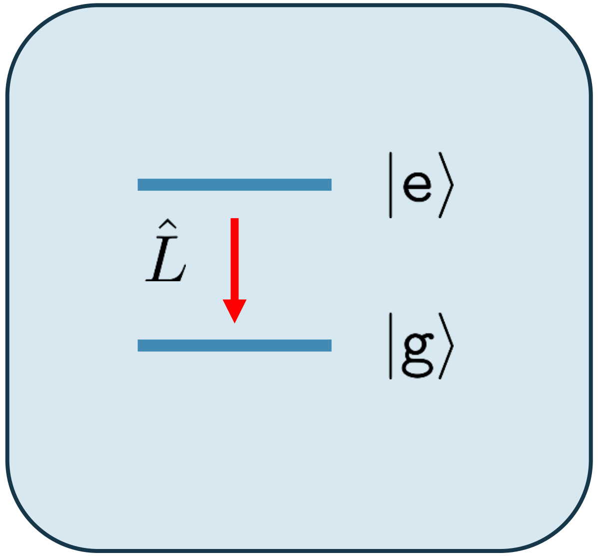
- Expand as a Dyson series and re-sum

$$|\Psi_t(\zeta)\rangle_S = |\phi\rangle_S + \langle\mathsf{e}|\phi\rangle_S \left((\lambda(t) - 1)|\mathsf{e}\rangle_S + \int_0^t \zeta_\tau \lambda(\tau)\, d\tau\, |\mathsf{g}\rangle_S \right),$$

where $\lambda(t) = \sum_{k=0}^{\infty} (-1)^k I_k(t)$ with $I_0(t) = 1$ and, for $k \geq 1$,

$$I_k(t) = \int_{t \geq \tau_{2k} > \tau_{2k-1} > \cdots > \tau_1 \geq 0} d\tau_{2k} \cdots d\tau_1\, K(\tau_{2k}, \tau_{2k-1}) \cdots K(\tau_2, \tau_1).$$

Exactly Solvable Model



- We note that

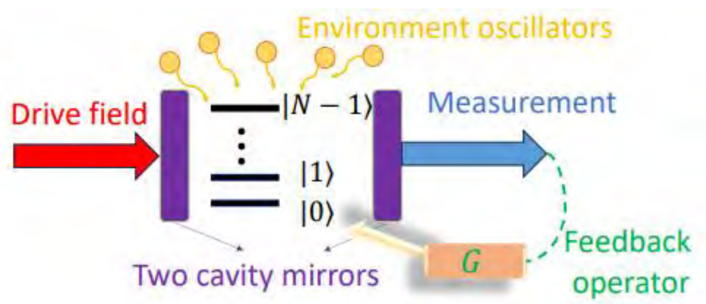
$$\dot{\lambda}(t) = -\int_0^t K(t,\tau)\lambda(\tau)\,d\tau.$$

- By substitution, the solution satisfies the Strunz-Diósi equation.
- For the "Markov" case we have

$$K(t,s) = \gamma\,\delta(t-s),$$

here we have $I_k(t) = \frac{1}{k!}(\gamma/2)^k$, and so $\lambda(t) = e^{-\gamma t/2}$.

Back to the Control Problem



- N-level atom version

$$\frac{\delta|\psi(t)\rangle}{\delta z_s} = \sum_{n=1}^{N-1} f_n(t,s) L_n |\psi(t)\rangle,$$

where

$$F_n(t) = \int_0^t \alpha(t,s) f_n(t,s) \mathrm{d}s$$

satisfies

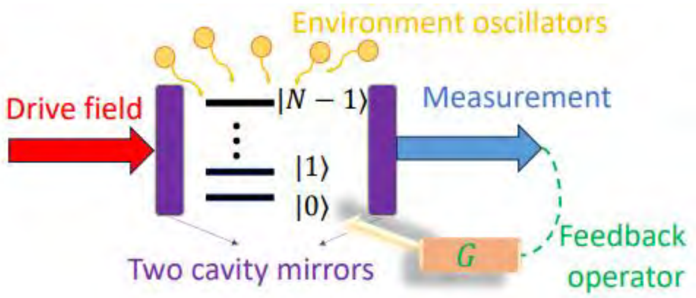
$$\frac{\mathrm{d}}{\mathrm{d}t} F_n(t) = \kappa_n F_n^2(t) - (\gamma + i\Omega - i\tilde{\omega}_n) F_n(t) + \frac{\gamma \chi_n}{2}$$

- The Markov case would lead to

$$F_n(t) = \int_0^t \delta(t-s) f_n(t,s) \mathrm{d}s = \frac{\chi_n}{2}$$

"Asymptotic Markovianity" is achieved for specific parameter regimes!

Back to the Control Problem



- Otherwise, linear equations for correlations, but with

  time-dependent coefficients

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_n^+\sigma_n^-\rangle &= -i\bar{g}_n\langle\sigma_n^+a\rangle + i\bar{g}_n^*\langle\sigma_n^-a^\dagger\rangle + i\bar{g}_{n+1}\langle\sigma_{n+1}^+a\rangle - i\bar{g}_{n+1}^*\langle\sigma_{n+1}^-a^\dagger\rangle \\
&\quad - \left(F_n(t)+F_n^*(t)\right)\kappa_n\langle\sigma_n^+\sigma_n^-\rangle + \left(F_{n+1}(t)+F_{n+1}^*(t)\right)\kappa_{n+1}\langle\sigma_{n+1}^+\sigma_{n+1}^-\rangle, \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_n^+a\rangle &= -i\Delta_n\langle\sigma_n^+a\rangle - i\bar{g}_n^*\langle\sigma_n^+\sigma_n^-\rangle + i\bar{g}_n^*\langle a^\dagger a\rangle - F_n^*(t)\kappa_n\langle\sigma_n^+a\rangle, \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_n^-a^\dagger\rangle &= i\Delta_n\langle\sigma_n^-a^\dagger\rangle + i\bar{g}_n\langle\sigma_n^+\sigma_n^-\rangle - i\bar{g}_n\langle a^\dagger a\rangle - F_n(t)\kappa_n\langle\sigma_n^-a^\dagger\rangle, \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle a^\dagger a\rangle &= i\sum_n\left(\bar{g}_n\langle\sigma_n^+a\rangle - \bar{g}_n^*\langle\sigma_n^-a^\dagger\rangle\right),
\end{aligned}
$$

Thank You

For Your Attention,